Formal Moduli Spaces for Modules over Planar Noncommutative Quadrics

by

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Contents

Preface

Given an associative k-algebra A for a field k, we want to give a local description of the moduli space of indecomposable left A-modules. The tool for this will be deformation theory and homological algebra. The goal of this thesis is to calculate formal moduli spaces of 2-dimensional indecomposable modules generated by two elements with a quadratic relation. We will also draw some conclusions about simplifyability of modules, via calculations of cycles in the induced extension graph.

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1. Introduction

A general goal of mathematical research is classification of objects [3]. Given a certain class of objects, we want to classify some objects within that class as being more similar in a particular way. This work often reveals hidden structures and relations of the studied objects, which again deepens our understanding of the subject.

When studying modules of rings in modern abstract algebra, it is a natural question to ask whether there is a convenient way of classifying the set of all modules belonging to a ring A. A way of doing this is to construct a space called the *moduli space* of A. This space can be thought of as a parameter set, where each point in the space corresponds to an equivalence class of indecomposable A-modules. The space itself often has an interesting geometrical structure, but unfortunately it can be very difficult and sometimes impossible to give a global characterization of it [6]. A usual approach is therefore to use the theory of deformations in a local study as an attempt to obtain information on the moduli space. In many ways, deformation theory is the calculus of moduli spaces describing all of its small perturbations [3].

A way to familiarize with these abstract notions is to exemplify theory by studying one type of ring and do calculations on it. This thesis reflects my attempt at this technique while approaching unfamiliar and abstract theory. The thesis is divided into two parts, one part presenting theory and the other exemplifying it by calculation.

We begin in **Section 2** by introducing the Hochschild cohomology induced by the Hochschild differential operator. The cup product and the commutator bracket will also be defined. In **Section 3** we will present the theory of deformations and deduce some properties of R-deformations. Both Section 2 and 3 are based on the work of Christoff Geiss in [2]. **Section 4** is a discussion of different ways of defining the Ext-group, and that these definitions produce the same result. This section is a study of theory presented in [7]. We continue in **Section 5** with a basic introduction to moduli spaces and simplifyable modules based on [5]. Here we will discuss a geometrical condition satisfied when dealing with simplifyable modules. **Section 6** contains various examples illustrating calculations of local moduli spaces in the affine noncommutative plane. We will also present the noncommutative partial differentiation [4], and how to use it to produce projective resolutions of free modules. An example involving the geometrical condition for simplifiable modules is displayed, before ending the thesis with a discussion of when there is a non-zero Ext between two modules over a general planar noncommutative quadric. Much of the work done in this thesis is inspired by the private lectures given by Arne B. Sletsjøe [6], in the Spring of 2011.

Notation. Let k be a closed field and A a Noetherian associative unitary k-algebra with underlying vector space k^n and ring multiplication $\alpha \in \mathrm{Hom}_k(A \otimes_k A, A)$. Let also M be an A-module with module operation $\mu \in \mathrm{Hom}_k(A \otimes_k M, M)$. When A is a polynomial ring with two indeterminates x, y and $M = k$ with μ defined as $x \mapsto a$ and $y \mapsto b$, we denote M with μ as $M = k(a, b)$.

2. Hochschild Cohomology

Given a k-algebra A over a field k with multiplication $\alpha \in \mathrm{Hom}_k(A \otimes_k A, A)$, and an A-bimodule M with multiplication $\mu \in \mathrm{Hom}_k(A \otimes_k M, M)$. We will often write $\alpha(a,b) := ab$ and $\mu(a,m) := am$ for $a, b \in A$ and $m \in M$. Define

$$A^{\otimes n} := \underbrace{A \otimes_k A \otimes_k \cdots \otimes_k A}_{n} \text{ and}$$
$$C^n := \mathrm{Hom}_k(A^{\otimes n}, M)$$

with the *composition product*

$$\circ : C^n \times C^m \to C^{(n+m-1)}$$
$$\beta \circ \gamma = \sum_{i=1}^{n} (-1)^{mi} \beta(a_0, \cdots, a_{i-2}, \gamma(a_{i-1}, a_i, \cdots, a_{i+m-1}), a_{i+m}, \cdots, a_{n+m-1}).$$

The graded *commutator bracket* is defined via this composition product by

$$[\beta, \gamma] := \beta \circ \gamma - (-1)^{|\beta||\gamma|} \gamma \circ \beta$$

for homogenous elements β and γ. The degree of $\beta \in C^n$ is $|\beta| = n - 1$. An immediate and useful result is the following lemma

Lemma 2.1. *The multiplication α of A is associative if and only if $\alpha \circ \alpha = 0$.*

Proof. $\alpha \circ \alpha$ takes $2 + 2 - 1 = 3$ elements of A. So let $a, b, c \in A$, then

$$\begin{aligned}(\alpha \circ \alpha)(a,b,c) &= -\alpha(a, \alpha(b,c)) + \alpha(\alpha(a,b), c) \\ &= -a(bc) + (ab)c\end{aligned}$$

which equals 0 if and only if α is associative. $\square$

Definition 2.1. *The Hochschild differential $\delta^n : C^n \to C^{n+1}$ with respect to α is defined by*

$$\begin{aligned}(\delta^n \phi)(a_0, \ldots, a_n) = a_0 \phi(a_1, \ldots, a_n) &+ \sum_{i=1}^{n} (-1)^i \phi(a_0, \cdots, a_{i-1}a_i, \cdots, a_n) \\ &+ (-1)^{n+1} \phi(a_0, \ldots, a_{n-1}) a_n.\end{aligned}$$

This expression is rather complicated, but we can simplify it by using the bracket notation. If we force the last term to have a negative sign, we can combine the first and the last term to be $\alpha \circ \phi$. The big middle sum then equals $\phi \circ \alpha$, so combined we get

$$\begin{aligned}(\delta^n \phi)(a_0, \ldots, a_n) &= a_0 \phi(a_1, \ldots, a_n) - \phi(a_0, \ldots, a_{n-1}) a_n \\ &\quad - (-1)^{|\phi|} \sum_{i=1}^{n} (-1)^i \phi(a_0, \cdots, a_{i-1}a_i, \cdots, a_n) \\ &= \alpha \circ \phi - (-1)^{|\phi|} \phi \circ \alpha \\ &= [\alpha, \phi].\end{aligned}$$

In this thesis we will only consider *graded Lie algebras* in which the Jacobi identity applies. We will state this as a lemma, even though it is considered an identity.

Lemma 2.2. *The Jacobi identity on graded Lie algebras is as follows*

$$[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{|\alpha||\beta|} [\beta, [\alpha, \gamma]]$$

From the Jacobi identity we immediately get the following result.

Lemma 2.3. *If α is associative and $char(k) \neq 2$, then $\delta^{i+1} \delta^i = 0$ for any $i \in \mathbb{Z}_+$.*

Proof. α is of degree 1, so the Jacobi identity yields

$$\begin{aligned}\delta^{i+1}\delta^i(\phi) &= \delta^{i+1}(\delta^i(\phi)) = [\alpha,[\alpha,\phi]] \\ &= [[\alpha,\alpha],\phi] + (-1)^1[\alpha,[\alpha,\phi]] = [0,\phi] - [\alpha,[\alpha,\phi]] = -[\alpha[\alpha,\phi]] \\ &\Rightarrow 2[\alpha,[\alpha,\phi]] = 0 \\ &\Rightarrow \delta^{i+1}\delta^i(\phi) = 0\end{aligned}$$

□

Since $\delta^{i+1}\delta^i = 0$, C^n can be characterized as cochains in a complex, denoted the *Hochschild complex.* Set

$$Z^i := \mathrm{Ker}(\delta^i)$$
$$B^i := \mathrm{Im}(\delta^{i-1}),$$

Define the cohomology of the Hochschild complex is as follows:

Definition 2.2. *The Hochschild cohomology is defined as*

$$HH^i = Z^i \big/ B^i$$

To further simplify notation we will later make use of the *cup product.* Let $\psi_1, \psi_2 \in \mathrm{Hom}_k(A \otimes M, M)$ and $a, b \in A$, then the cup product is defined by

$$(\psi_1 \smile \psi_2)(a,b) = \psi_1\big(a, \psi_2(b,-)\big),$$

Given these four elements, the cup product $(\psi_1 \smile \psi_2)(a,b)$ is a homomorphism from M to M.

3. Deformation Theory

Let R be in the category ℓ of punctured Artin rings [1]. An Artin ring R is *punctured* if there is a morphism $\pi : R \to k$ such that the inclusion morphism $k \hookrightarrow R$ composed with π is the identity on k. A morphism $\psi \in \mathrm{Mor}(\ell)$ of punctured rings is a k-algebra map $\psi : R \to S$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, "\pi"'] & S \arrow[dl, "\pi'"] \\ k & \end{tikzcd}$$

Fixing A and M, the deformation functor $\mathrm{Def}_M : \ell \to \text{sets}$ is defined as

$$\mathrm{Def}_M(R) = \big\{M_R \in A \otimes R\text{-mod} \mid M_R \otimes_\pi k \cong M \text{ and } M_R \text{ is flat over } R\big\}\big/ \sim,$$

where $M_R \sim M'_R$ if there exists a commutative diagram:

$$\begin{tikzcd} M_R \arrow[r, "\cong"] \arrow[d, "(-\otimes_\pi k)"'] & M'_R \arrow[dl, "(-\otimes_{\pi'} k)"] \\ M & \end{tikzcd}$$

Examples of rings satisfying these conditions are the *truncated polynomial rings* $R = k[t]/(t^n)$ for $n \geq 2$. For $n = 2$ we write $R = k[\epsilon]$. In both cases the puncture morphism π is simply $t \mapsto 0$ and the identity on M.

Definition 3.1. *Let $R \in \ell$. An R-deformation of μ is an element*

$$\mu_R \in Hom_R((A \otimes_k R) \otimes_R (M \otimes_k R), (M \otimes_k R))$$

which reduces modulo R to μ and is associative, i.e.

$$\mu_R\big(a \otimes_R \mu(b,m)\big) = \mu_R\big(\alpha(a,b) \otimes_R m\big) \text{ for } a, b \in A, m \in M \tag{3.1}$$

If $R = k[\epsilon]$ we say that μ_R is an *infinitesimal deformation*. When A and M are given, the goal of deformation theory is to classify the set of all R-deformations of the A-module M. If we take the inverse limit of a sequence of the Artin rings $k[t]/(t^n)$, we get the *formal polynomial ring* $R = k[[t]]$ yielding a *formal deformation*. R is then a pro-Artin ring containing the information of all the truncated rings, and is hence the classification we seek.

We say that an infinitesimal deformation $\mu_{k[\epsilon]}$ of μ is *integrable* if there exists a formal deformation $\mu_{k[[t]]}$ of μ which reduces via the projection $p_{t,\epsilon} : k[[t]] \to k[\epsilon]$ to $\mu_{k[\epsilon]}$. We will come back to integrable deformations and in which circumstances they occur in section 5.

Let $R = k[t]/(t^{n+1})$, $S = k[t]/(t^n)$, $M_S \in \mathrm{Def}_M(S)$ a deformation and a map $\psi : R \twoheadrightarrow S$. We say that ψ is a *small surjection morphism* in ℓ. In deformation theory we need an existence of a lifting from S to R. The following lemma gives us this lifting, however with an *obstruction* telling us where the lifting doesn't behave as wanted.

Lemma 3.1. *Given R, S, M_S and ψ as above, there exists a canonical obstruction*

$$o(\psi, M_S) \in Ext^2_A(M, M)$$

such that $o(\psi, M_S) = 0$ if and only if there exists a deformation $M_R \in Def_M(R)$ lifting M_S. If this is the case, the set of deformations in $Def_M(R)$ is in a bijective correspondance with the k-vector space $Ext^1_A(M, M)$.

Proof. See [1, prop. 5.1]. □

[1] Not necessarily commutative.

We have not yet discussed the Ext-functor, but we will keep in mind that the obstructions are located in the k-vector space $\mathrm{Ext}_A^2(M, M)$.

3.1. **Properties of formal deformations.**

As the formal deformations classifies all R-deformations with $R = k[t]/(t^n)$, we can study properties of the formal deformations and treat the rest as special cases. Let $R = k[[t]] \cong k \oplus kt \oplus kt^2 \oplus \cdots$ and μ_R be an R-deformation. Then

$$\mu_R : \big(A \otimes_k k[[t]]\big) \otimes_R \big(M \otimes_k k[[t]]\big) \to \big(M \otimes_k k[[t]]\big)$$

and by R-linearity we get:

$$\mu_R(a, m) = am + t\psi_1(a, m) + t^2\psi_2(a, m) + \cdots \tag{3.2}$$

for a family of k-linear maps $\psi_i : \mathrm{Hom}_k(A \otimes M, M)$. This indicates that μ_R is determined by $\{\psi_i\}$ and μ. Because we know that μ_R satisfies condition (3.1), we can use this and (3.2) to deduce some properties of $\{\psi_i\}$. Calculating the left side of (3.1), we get

$$\begin{aligned}
&\mu_R(a, \mu_R(b, m)) \\
&= a\mu_R(b, m) + t\psi_1(a, \mu_R(b, m)) + t^2\psi_2(a, \mu_R(b, m)) + \cdots \\
&= a(bm) + ta\psi_1(b, m) + t^2 a\psi_2(b, m) + t^3 a\psi_3(b, m) + \cdots \\
&\quad + t\psi_1(a, bm) + t^2\psi_1(a, \psi_1(b, m)) + t^3\psi_1(a, \psi_2(b, m) + \cdots \\
&\quad + t^2\psi_2(a, bm) + t^3\psi_2(a, \psi_1(b, m)) + t^4\psi_2(a, \psi_2(b, m)) + \cdots ,
\end{aligned}$$

whereas the right side yields

$$\begin{aligned}
&\mu_R(\alpha(a, b), m) = \mu_R(ab, m) \\
&= (ab)m + t\psi_1(ab, m) + t^2\psi_2(ab, m) + t^3\psi_3(ab, m) + \cdots .
\end{aligned}$$

Sorting the terms by the degree of t, we get a set of equations describing the properties of $\{\psi_i\}$:

- $(ab)m = a(bm)$
- $a\psi_1(b, m) - \psi_1(ab, m) + \psi_1(a, bm) = 0$
- $a\psi_2(b, m) - \psi_2(ab, m) + \psi_2(a, bm) = -\psi_1(a, \psi_1(b, m))$
- $a\psi_3(b, m) - \psi_3(ab, m) + \psi_3(a, bm) = -\psi_1(a, \psi_2(b, m)) - \psi_2(a, \psi_1(b, m))$

  $\vdots \qquad\qquad\qquad\qquad \vdots$

By omitting m, using the Hochshild differential and the previously defined cup product, we can simplify these equations as follows:

- $(ab)m = a(bm)$
- $\delta^1\psi_1(a, b) = 0$
- $\delta^1\psi_2(a, b) = -(\psi_1 \smile \psi_1)(a, b)$
- $\delta^1\psi_3(a, b) = -(\psi_1 \smile \psi_2)(a, b) - (\psi_2 \smile \psi_1)(a, b)$

  $\vdots \qquad\qquad \vdots$

- $\delta^1\psi_n(a, b) = -\sum_{i+j=n} \psi_i \smile \psi_j(a, b)$

  $\vdots \qquad\qquad \vdots$

This means that we should be able to find μ_R by successively calculating ψ_i and using these formulae. However, we must find out whether there are obstructions to doing this in Ext_A^2 or not. Even if we know that these relations exists, no procedure for finding $\{\psi_i\}$ is apparent. First of all, there are infinitely many terms to calculate, and second it is sometimes very difficult or even impossible to do so. We can simplify the first problem by studying the infinitesimal deformations $R = k[\epsilon]$, and using the lifting property to proceed with studying

$R = k[t]/(t^3)$ and so on. In many ways, this procedure resembles Taylor expansion with higher degree of accuracy the higher power of t we calculate. We shall later see that if there are no obstructions in the liftings $\big(\mathrm{Ext}^2_A(M,M) \neq 0\big)$, every infinitesimal deformation is integrable. This basically means that it suffices to calculate the infinitesimal deformation to get all information about the tangent space of M.

In the next chapter we will introduce Ext to provide us tools to give more thorough answers to these questions.

4. Ext

In this section we will look at various ways of defining the Ext-group and its relation to the deformation of a module. The Ext-group, $\mathrm{Ext}_A(M,N)$, was originally defined as the equivalence classes of *extensions of M by N* [6].

4.1. Derivations, Extensions and Ext.

Definition 4.1. *An extension ξ of M by N for A-modules M and N, is an exact sequence $0 \to N \to E \to M \to 0$. Two extensions ξ and ξ' are equivalent if there is a commuting diagram of A-module homomorphisms:*

$$\begin{tikzcd}
\xi & 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, equal] & 0 \\
\xi' & 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Finally, ξ is a split extension if $E = N \oplus M$ as A-modules.

So an element of Ext is the equivalence class of an extension. Ext emerges in many areas of homological algebra, and it can be difficult to understand what this concept really means. To expand our understanding of Ext and what it really is, we shall in this subsection derive properties of an extension and its equivalence class.

Another way to define Ext is by *derivations*. The set of derivations of A is defined as:

$$\mathrm{Der}\big(A, \mathrm{Hom}_k(M,N)\big) := \big\{D : A \to A \in \mathrm{Hom}_k(M,N) \mid D(ab)(m) = aD(b)(m) + D(a)b(m)\big\},$$

for elements $a, b \in A$ and $m \in M$.

Let ξ be an extension of M by N with A-module homomorphisms f and g as in the above diagram. As f is injective and g is surjective, it will be convenient to understand E as a direct sum $N \oplus M$ with some equivalence relation identifying suitable elements. Because f is injective and g is surjective, we write $f(n) = (n,0)$ and $g(n,m) = m$ for $n \in N$ and $m \in M$. We use the A-module properties of E and the A-module homomorphism properties of f and g to explore the structure of ξ:

(1) $f(an) = af(n)$ for $a \in A$ and $n \in N$,
(2) $g(a(n,m)) = ag(n,m)$ for $m \in M$,

which implies

(1) $(an,0) = a(n,0) \in N \oplus M$
(2) $g(a(n,m)) = g(n',m') = m'$ and $ag(n,m) = am$
$\Rightarrow am = m'$

By using the above identities we can now calculate

$$
\begin{aligned}
a(n,m) &= a\big((n,0)+(0,m)\big) = a(n,0)+a(0,m) = (an,0)+(\gamma(a,m),am) \\
&= \big(an+\gamma(a,m),am\big),
\end{aligned}
$$

where $\gamma(a,m) \in N$ depending on a and m. Another requirement from the module structure of E is:

$$
\begin{aligned}
(ab)(n,m) &= a(b(n,m)) \\
\Rightarrow (abn+\gamma(ab,m),abm) &= (abn+a\gamma(b,m)+\gamma(a,bm),abm) \\
\Rightarrow abn+\gamma(ab,m) &= abn+a\gamma(b,m)+\gamma(a,bm) \\
\Rightarrow \gamma(ab,m) &= a\gamma(b,m)+\gamma(a,bm)
\end{aligned}
$$

By observing that γ is behaving as a derivation, we define

$$D : A \to \mathrm{Hom}_k(M,N) \text{ by letting } D(a)(m) = \gamma(a,m)$$

and observe that

$$
\begin{aligned}
&D(ab)(m) = aD(b)(m) + D(a)b(m) \\
\Rightarrow &D \in \mathrm{Der}\big(A, \mathrm{Hom}_k(M,N)\big)
\end{aligned}
$$

By investigating the properties of ξ, we found that there is a derivation belonging to ξ. There are many types of derivations. If we take an homomorphism $\beta \in \mathrm{Hom}_k(M,N)$, we can construct an *inner derivation* by the commutator $[-,\beta]$. The set of inner derivations is defined as

$$\mathrm{Inder}_k(A, \mathrm{Hom}_k(M,N)) := \{D_\beta \in \mathrm{Der}(A) \mid D_\beta = [-,\beta],\ \beta \in \mathrm{Hom}_k(M,N)\}.$$

It is easy to check that an inner derivation D_β is in fact a derivation by:

$$
\begin{aligned}
D_\beta(ab) &= [ab,\beta] = ab\beta - \beta ab = ab\beta + (a\beta b - a\beta b) - \beta ab \\
&= a(b\beta - \beta b) + (a\beta - \beta a)b = a[b,\beta] + [a,\beta]b \\
&= aD_\beta(b) + D_\beta(a)b.
\end{aligned}
$$

In Ext, we know that some extensions are equivalent, so to get a definition for Ext by derivations we must find a type of derivation yielding the same equivalence. Consider again the commuting diagram of two equivalent extensions ξ and ξ':

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, "\psi"', "\cong"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

from which we can deduce the following relations:

$$
\begin{aligned}
g(\psi(0,m)) &= g(0,m) = m \\
\psi(0,m) &= (\beta(m),m) \quad \text{ for some } \beta \in \mathrm{Hom}_k(M,N) \\
\psi(n,0) &= \psi(f(n)) = f(n) = (n,0) \\
\Rightarrow \psi(n,m) &= \psi(n,0) + \psi(0,m) = \psi(f(n)) + \psi(0,m) \\
&= f(n) + \psi(0,m) = (n,0) + \psi(0,m) \\
&= (n,0) + (\beta(m),m) = (n+\beta(m),m).
\end{aligned}
$$

These identities, D and the homomorphism property of ψ combined yields

$$
\begin{aligned}
\psi(a(0,m)) &= a\psi(0,m) \\
\psi\big(D(a)(m),am\big) &= a\big(\beta(m),m\big) \\
\big(D(a)(m)+\beta(am),am\big) &= \big(a\beta(m)+D'(a)(m),am\big)
\end{aligned}
$$

where $D'(a)(m) := D(a)\big(\beta(m)\big)$. This implies that

$$D(a)(m) + \beta(am) = a\beta(m) + D'(a)(m)$$
$$\Rightarrow D(a)(m) - D'(a)(m) = a\beta(m) - \beta(am)$$

Using the bracket notation introduced in the previous chapter, we get

$$\Rightarrow D(a) - D'(a) = a\beta - \beta a = [a, \beta]$$
$$\Rightarrow D - D' = [-, \beta] = D_\beta \in \mathrm{Inder}_k(A, \mathrm{Hom}_k(M, N)).$$

By studying the properties of two equivalent extensions, we found that ξ' has an inner derivation. We can now give a definition of the Ext-group of M by N in terms of derivations:

$$\mathrm{Ext}_A(M, N)) = \mathrm{Der}_k(A, \mathrm{Hom}_k(M, N)) \;\big/\; \mathrm{Inder}_k(A, \mathrm{Hom}_k(M, N)).$$

Later work in this field has shown that the Ext-group also emerges in other areas, giving rise to other more general defeninitions [6]. In the next subchapter we will have a look at these.

4.2. **The Ext-group defined by resolutions.**

An algebraic resolution is a way of describing the structure of a module. There are two important types of resolutions we use to define the *Ext-functor*, namely projective and injective resolutions.

Definition 4.2. *Let P be an A-module. We say that P is projective if for every map $P \to M$ and surjection $E \to M$ there exists at least one map $P \to E$ such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[d] \arrow[dr, dotted, "\exists"] & \\ M & E \arrow[l, two heads] \end{tikzcd}$$

Definition 4.3. *A projective resolution of an A-module M is an exact seqence*

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

where each P_i is a projective A-module for every $i \in \mathbb{N}$.

Applying the functor $\mathrm{Hom}_A(-, N)$ to a projective resolution and we obtain the following chain complex

$$\mathrm{Hom}_A(P_0, N) \xrightarrow{d^0} \mathrm{Hom}_A(P_1, N) \xrightarrow{d^1} \mathrm{Hom}_A(P_2, N) \longrightarrow \cdots$$

Definition 4.4. *The Ext-functor of projective resolutions is the cohomology of the previously defined chain complex. This means explicitly*

$$Ext^i_A(M, N)_P = Ker(d^i)/Im(d^{i-1}) \text{ for } i \geq 1$$

A convenient alternative way of viewing projective resolutions is apparent in the following lemma:

Lemma 4.1. *An A-module is projective if and only if it is a direct summand of a free A-module.*

Proof. See [7, page 33]. ☐

As a dual point of view, we will also define Ext via *injective resolutions* .

Definition 4.5. *Let I be an A-module. We say that I is injective if for every map $N \to I$ and injection $N \hookrightarrow E$ there exists at least one map $E \to I$ such that the following diagram commutes:*

$$\begin{tikzcd} I & \\ N \arrow[u] \arrow[r, hook] & E \arrow[ul, dotted, "\exists"'] \end{tikzcd}$$

Definition 4.6. *An injective resolution of an A-module N is an exact seqence*

$$0 \longrightarrow N \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

where each I_i is an injective A-module for every $i \geq 0$.

Use the functor $\mathrm{Hom}_A(M, -)$ on an injective resolution and we obtain the following chain complex

$$\mathrm{Hom}_A(M, I^0) \xrightarrow{d_0} \mathrm{Hom}_A(M, I^1) \xrightarrow{d_1} \mathrm{Hom}_A(M, I^2) \longrightarrow \cdots$$

This leads us to another definition of Ext:

Definition 4.7. *The Ext-functor of injective resolutions is the homology of this chain complex. That is*

$$Ext^i_A(M, N)_I = Ker(d_i)/Im(d_{i-1}) \text{ for } i \geq 1$$

Note that the Ext-functor yields, not only one Ext-group as before, but several numbered from 1 to possibly infinitely many depending on the given resolution of M or N. In the examples provided in this thesis we only look at Noetherian rings yielding finite length of their resolutions [7, Prop. 4.1.5]. We shall later see that this way of defining the Ext-functor yields $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}_A(M, N)$.

The two ways of defining the Ext-functor look very similar, and we shall prove that they in fact produce isomorphic results. Given an extension

$$\xi : \quad 0 \to N \to E \to M \to 0,$$

a projective resolution of M and an injective resolution of N, consider the following diagram obtained by successively applying the functors $\mathrm{Hom}_A(M, -)$, $\mathrm{Hom}_A(P_0, -)$, $\mathrm{Hom}_A(P_1, -)$, $\cdots$ on the injective resolution of N. The result is a commutative diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
0 \arrow[r] & \mathrm{Hom}_A(M, I_2) \arrow[r, "s_0"] \arrow[u] & \mathrm{Hom}_A(P_0, I_2) \arrow[r, "s_1"] \arrow[u] & \mathrm{Hom}_A(P_1, I_2) \arrow[r] \arrow[u] & \iddots & \\
0 \arrow[r] & \mathrm{Hom}_A(M, I_1) \arrow[r, "r_0"] \arrow[u, "d_1"] & \mathrm{Hom}_A(P_0, I_1) \arrow[r, "r_1"] \arrow[u, "j_2"] & \mathrm{Hom}_A(P_1, I_1) \arrow[r] \arrow[u] & \mathrm{Hom}_A(P_2, I_1) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(M, I_0) \arrow[r, "q_0"] \arrow[u, "d_0"] & \mathrm{Hom}_A(P_0, I_0) \arrow[r, "q_1"] \arrow[u, "j_1"] & \mathrm{Hom}_A(P_1, I_0) \arrow[r, "q_2"] \arrow[u, "i_1"] & \mathrm{Hom}_A(P_2, I_0) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(M, N) \arrow[r] \arrow[u] & \mathrm{Hom}_A(P_0, N) \arrow[r, "d^0"] \arrow[u, "j_0"] & \mathrm{Hom}_A(P_1, N) \arrow[r, "d^1"] \arrow[u, "i_0"] & \mathrm{Hom}_A(P_2, N) \arrow[r] \arrow[u, "k_0"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In the diagram above, every row except the bottom one, and every column except the left one, is exact. This is true because I_i and P_j respectively are injective and projective.

Lemma 4.2. *Given two A-modules M and N and an extension of M by N, then $Ker(d^1)/Im(d^0) \cong Ker(d_1)/Im(d_0)$ implying that $Ext^1_A(M, N)_P \cong Ext^1_A(M, N)_I$.*

Proof. Considering the above diagram, our goal is to find a well defined morphism $\mathrm{Ker}(d^1) \to \mathrm{Ker}(d_1)$ and $\mathrm{Im}(d^0) \to \mathrm{Im}(d_0)$. Let $x \in \mathrm{Ker}(d^1)$. Then

$$\begin{aligned}
&(k_0 \circ d^1)(x) = 0 \\
\Rightarrow &(q_2 \circ i_0)(x) = 0 \\
\Rightarrow &i_0(x) \in \mathrm{Ker}(q_2) = \mathrm{Im}(q_1) \\
\Rightarrow &\exists y \in \mathrm{Hom}_A(P_0, I_0) : q_1(y) = i_0(x)
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&(i_1 \circ i_0)(x) = 0 \\
\Rightarrow &j_1(y) \in \mathrm{Ker}(r_1) = \mathrm{Im}(r_0) \\
\Rightarrow &\exists z \in \mathrm{Hom}_A(M, I_1) : r_0(z) = j_1(y)
\end{aligned}$$

Also $(j_2 \circ j_1)(y) = 0$ which implies that $(s_0 \circ d_1)(z) = (j_2 \circ r_0)(z) = (j_2 \circ j_1)(y) = 0$. But s_0 is injective, so $f_1(z) = 0$. We've now got a well defined morphism $\mathrm{Ker}(d^1) \to \mathrm{Ker}(d_1)$.
Let now $x \in \mathrm{Im}(d^0)$ and $d^0(y) = x$. Because both j_0 and i_0 are injective we write $i_0(x) = x$ and $j_0(y) = y$. From the diagram we see that $q_1(y) = x$. Let z be another element with $q_1(z) = x$ *(Remark: If no such z exists, q_1 is injective and $Hom_A(M, I_0) = 0$, collapsing the diagram).* Now $z - y$ is an element satisfying

$$\begin{aligned}
q_1(z - y) = q_1(z) - q_1(y) = x - x = 0 \\
\Rightarrow (z - y) \in \mathrm{Ker}(q_1) = \mathrm{Im}(q_0).
\end{aligned}$$

The map q_0 is injective so again we simplify by denoting $q_0(z'') = z - y$. From the diagram we see that $j_1(y) = (j_1 \circ j_0)(y) = 0$, which indicates

$$(r_0 \circ d_0)(z'') = j_1(z'') = j_1(z) - 0 = j_1(z) := z'.$$

The fact that r_0 is injective implies $f_0(z - y) = z' \in \mathrm{Im}(d_0)$. Now we have a well defined mapping from $\mathrm{Im}(d^0)$ to $\mathrm{Im}(d_0)$. This mapping is obviously an isomorphism so we can now conclude

$$\mathrm{Ext}^1_A(M, N) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong \mathrm{Ker}(d_1)/\mathrm{Im}(d_0)$$

$\square$

Although we only treat $\mathrm{Ext}^1_A(M, N)$ in this lemma, a more abstract approach will prove the same result for $\mathrm{Ext}^i_A(M, N)$ for $i \geq 1$ [7, Thm. 2.7.6].

4.3. $\mathbf{Ext}^1_A(M, N)$ and $\mathbf{Ext}_A(M, N)$.

In this section we will prove that there is a one-to-one correspondence between the equivalence classes of extensions of N by M and elements in $\mathrm{Ext}^1_A(M, N)$ defined by resolutions. To do this we need a couple of definitions and lemmata.

Definition 4.8. *The pushout of two A-module homomorphisms f and g with a common domain Z consists of an A-module P and two A-module homomorphisms $i_1 : X \to P$ and $i_2 : Y \to P$ for which the following diagram commutes:*

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "i_1"] \\
Y \arrow[r, "i_2"] & P
\end{tikzcd}$$

Moreover the diagram is universal such that if there exists another A-module Q and A-module homomorphisms j_1 and j_2 for which the diagram also commutes, there exist a unique A-module *homomorphism h such that the following diagram commutes:*

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\
Y \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & P \arrow[dr, dotted, "\exists! h"] & \\
 & & Q
\end{tikzcd}$$

The definition of a pushout can also be generalized to other categories. For example, a pushout in set theory is the union of two sets. If $Z = X \cap Y$ and f and g are inclusions, then $P = X \cup Y$ with the equivalence relation $x \sim y$ if $x \in Z$ and $y \in Z$. It is convenient to think of the pushout P as a sum of X and Y divided out by the equivalence relation $\sim$. For the main result of this section we will need the next lemma:

Lemma 4.3. *For a resolution $0 \to K \to P_0 \to M \to 0$ of A-modules where P_0 is projective, and for every A-module N we have the following exact sequence:*

$$0 \to Hom_A(M, N) \to Hom_A(P_0, N) \to Hom_A(K, N) \to Ext_A^1(M, N) \to 0$$

Proof. Consider the following diagram where $P \to M$ is a projective resolution of M and every straight sequence is exact:

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dr] & 0 & & & \\
 & K_2 \arrow[dr] & & K \arrow[u] \arrow[dr] & & & \\
 & \cdots \arrow[r] & P_2 \arrow[r] \arrow[dr] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[dr] & M \arrow[r] \arrow[d, equal] & 0 \\
 & & & K_1 \arrow[u] \arrow[dr] & & M \arrow[dr] & \\
 & & & 0 \arrow[u] & 0 & & 0
\end{tikzcd}$$

Applying $\mathrm{Hom}_A(-, N)$ to this diagram, we get

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_A(M, N) \arrow[r] & \mathrm{Hom}_A(P_0, N) \arrow[r, "p"] \arrow[dr, "d_0"] & \mathrm{Hom}_A(K, N) \arrow[r, "\partial"] \arrow[d, "q_1"] & \mathrm{Ext}_A^1(M, N) \arrow[r] & 0 \\
 & & & \mathrm{Hom}_A(P_1, N) \arrow[ur, two heads, "r"] \arrow[d, "q_2"] \arrow[dr, "d_1"] & \mathrm{Ker}(d_1) \arrow[l, hook'] \arrow[u, two heads] & \\
 & & 0 \arrow[r] & \mathrm{Hom}_A(K_1, N) \arrow[r, "q_3"] & \mathrm{Hom}_A(P_2, N) &
\end{tikzcd}$$

where the upper horizontal sequence arises from the upper right diagonal sequence of the first diagram.

The kernel of d_1 maps injectively to $\mathrm{Hom}_A(P_1, N)$ and surjectively to $\mathrm{Ker}(d_1)/\mathrm{Im}(d_0) = \mathrm{Ext}_A^1(M, N)$.

Let $z \in \mathrm{Ext}^1_A(M,N)$. Then $z \in \mathrm{Ker}(d_1) \Rightarrow d_1(z) = 0$. Because q_3 is injective, it follows that $q_2(z) = 0$ implying $z \in \mathrm{Im}(q_1)$ and $\exists x \in \mathrm{Hom}_A(K,N)$ such that $q_1(x) = z$.
Because $z \in \mathrm{Ker}(d_1) = \mathrm{Im}(d_0)$ there $\exists y \in \mathrm{Hom}_A(P_0,N)$ such that $d_0(y) = z$.

$$\begin{aligned} &\Rightarrow (q_1 \circ p)(y) = z \Rightarrow p(y) = x \text{ (because } q_1 \text{ is injective)} \\ &\Rightarrow x \in \mathrm{Im}(p) \end{aligned}$$

Let $r : \mathrm{Hom}_A P_1, N) \to \mathrm{Ext}(M,N)$ which must be surjective, and let $\partial = r \circ q_1$. We are now left to show that $\partial(x) = 0$. Since $\mathrm{Ext}^1_A(M,N) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_0)$, we see that

$$\begin{aligned} \partial(x) = \partial(p(y)) = r\Big(q_1\big(p(y)\big)\Big) = r(d_0(y)) = 0 \\ \Rightarrow \mathrm{Im}(p) = \mathrm{Ker}(\partial) \end{aligned}$$

which proves the desired result. □

This lemma tells us that $\mathrm{Ext}^1_A(M,N)$ gives us a "measure" of how $\mathrm{Hom}(-,N)$ fails to be exact. Keeping in mind the original notion of the Ext-group, the next lemma is of importance.

Lemma 4.4. *$Ext^1_A(M,N) = 0$ if and only if every extension of M by N is a split extension.*

Proof. Let ξ denote an extension sequence $0 \to N \to E \xrightarrow{q} M \to 0$. By applying the left exact functor $\mathrm{Hom}_A(M,-)$ on ξ, we get the following exact sequence:

$$\mathrm{Hom}_A(M,E) \xrightarrow{q_*} \mathrm{Hom}_A(M,M) \xrightarrow{\partial} \mathrm{Ext}^1(M,N)$$

If ξ splits ($E = N \oplus M$), then there exists a $\sigma \in \mathrm{Hom}_A(M,E)$ such that $q_*(\sigma) = \mathrm{id}_M$. By exactness it follows that $\partial(\mathrm{id}_M) = 0$, which means that $\mathrm{Ext}^1(M,N) = 0$.
Conversely, if $\mathrm{Ext}^1(M,N) = 0$, then q_* is surjective. This means that there exists a $\sigma \in \mathrm{Hom}_A(M,E)$ such that $q_*(\sigma) = \mathrm{id}_M$. Because the image of N is mapped to 0 by q, $\mathrm{Im}(\sigma) \cong M \Rightarrow \xi$ splits.

$$\begin{tikzcd} \xi : & 0 \arrow[r] & N \arrow[r] & E \arrow[r, "q"'] & M \arrow[r] \arrow[l, bend right, dotted, "\sigma"'] & 0 \end{tikzcd}$$

□

We now define $\Theta : \{\xi\} \to \partial(\mathrm{id}_M)$ and restate the lemma as follows:

$$\xi \text{ splits} \quad \Longleftrightarrow \quad \Theta(\xi) = 0 \text{ in } \mathrm{Ext}^1_A(M,N).$$

Theorem 4.1. *There is a one-to-one correspondance between the equivalence classes of extensions of M by N and $Ext^1_A(M,N)$ via the mapping of Θ.*

Proof. Choose a truncated projective resolution of M; $0 \to K \xrightarrow{j} P \to M \to 0$. We use lemma (4.3) to produce the following exact sequence

$$\mathrm{Hom}_A(P,N) \to \mathrm{Hom}_A(K,N) \xrightarrow{\partial} \mathrm{Ext}^1(M,N) \to 0$$

For an $x \in \mathrm{Ext}^1_A(M,N)$ we can now choose $\beta_x \in \mathrm{Hom}_A(K,N)$ with $\beta_x \mapsto x$. Let (E,σ,f) be the pushout of β_x and j such that $k \in K \mapsto (-\beta_x(k), j(k)) \in E$, then we can form the following diagram:

$$\begin{tikzcd} & 0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "\beta_x"] & P \arrow[r] \arrow[d, "\sigma"] & M \arrow[r] \arrow[d, equal] & 0 \\ \xi : & 0 \arrow[r] & N \arrow[r, "f"] & E \arrow[r, "g"] & M \arrow[r] & 0 \end{tikzcd}$$

where g is induced by $N \xrightarrow{0} M$ and the map $P \to M$. We are now left to prove that ξ is exact. Let $n \in \mathrm{Ker}(f)$, then:

$$\begin{aligned}
&f(n) = (n,0) = (0,0)\\
\Rightarrow&\exists\ y \in K \text{ with } \beta_x(y) = n \text{ and } j(y) = 0\\
\Rightarrow&j \text{ injective gives } y = 0 \text{ and } n = \beta_x(0) = 0\\
\Rightarrow&f \text{ is injective.}
\end{aligned}$$

Furthermore, $(n,p) \in \mathrm{Ker}(g) \Rightarrow p = j(y)$ for some $y \in K$, which means that

$$\begin{aligned}
(n,p) = (n,p) + (0,0) =&\big(n, j(y)\big) + \big(\beta_x(y), -(j(y)\big) = \big(n + \beta_x(y), 0\big)\\
\Rightarrow&\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)
\end{aligned}$$

Let $(n,p) \in \mathrm{Im}(f)$,

$$\begin{aligned}
\Rightarrow&\exists\ n' \in N \text{ such that } f(n') = (n,p) = (n',0)\\
\Rightarrow&g(n,p) = g(n',0) = 0\\
\Rightarrow&\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)\\
\Rightarrow&\mathrm{Im}(f) = \mathrm{Ker}(g).
\end{aligned}$$

Finally g is surjective because for every $m \in M$ there is a $p \in P$; $P \mapsto m$ with $g(\sigma(p)) = \mathrm{id}_M(m) = m$, implying that ξ is exact and hence an extension of M by N. By the naturality of ∂ we see that $\Theta(\xi) = x$ and that Θ is surjective.
If $\beta'_x \in \mathrm{Hom}_A(K,N)$ is another lift of x, then $\beta_x - \beta'_x \in \mathrm{Ker}(\partial)$ so there is an $i \in \mathrm{Hom}_A(P,N)$ with $\beta'_x = \beta_x + ij$. If E' is the pushout of j and β'_x, then the map $\psi : E \to E'$ with $\psi(n,p) = (n - i(p), p)$ is an isomorphism. ψ is well defined because

$$\begin{aligned}
\psi(0,0) = \psi(\beta_x(k), -j(k)) &= (\beta(k) + ij(k), -j(k)\\
&= (\beta'_x(k), -j(k)) = (0,0).
\end{aligned}$$

It is injective since

$$\begin{aligned}
(0,0) = \psi(n,p) &= (n - i(p), p)\\
&\Rightarrow (\beta'(k), -j(k)) = (n - i(p), p) \text{ for some } k \in K\\
&\Rightarrow p = -j(k)\\
&\Rightarrow n - i(p) = n + ij(k) = \beta'_x(k)\\
&\Rightarrow n = \beta'_x(k) - ij(k) = \beta_x(k)\\
&\Rightarrow (n,p) = (\beta_x(k), -j(k)) = (0,0)
\end{aligned}$$

and it is obviously surjective for $(n,p) = \psi(n+i(p), p)$. This shows that there is an equivalence between ξ and the exact sequence induced by β'_x. In fact, this construction gives therefore a set map between $\mathrm{Ext}^1_A(M,N)$ and the equivalence classes of extensions of M by N.
Conversely, given an extension ξ of M by N, the lifting property of P gives a map $\tau : P \to E$ and hence a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "\gamma"] & P \arrow[r] \arrow[d, "\tau"] & M \arrow[r] \arrow[d, equal] & 0\\
\xi: & 0 \arrow[r] & N \arrow[r, "i"] & E \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

Now E is the pushout of j and γ. Hence Θ is injective. $\square$

5. Moduli Spaces and Simplifyable Modules

The main goal is to find the space containing all *indecomposable modules* of A, called the *moduli space* of A. In this space every point represents a module which either is simple or a *non-splitting extension module*. By this we mean an A-module E in between two other A-modules N and M in a non-splitting extension, $\xi : 0 \to N \to E \to M \to 0$. The *decomposable modules*, modules that can be written as the sum of two non-zero submodules, are not of interest because they don't contain any additional information about the modules of A. The moduli space is in most cases very hard or even impossible to express globally, but deformation theory is a useful tool to explore the moduli space locally.

5.1. Moduli Spaces.

By choosing an A-module M, we use deformation theory to obtain a local tangent space giving us information about possible directions to move. For example the local moduli space of $A = k\langle x, y\rangle/(xy)$ in $M = k(0,0)$ is two dimensional ($\mathrm{Ext}^1_A(M, M) \cong k^2$), because we can move in the $x = 0$ and $y = 0$ direction to find other A-modules[2]. An important remark here, is that even though the local moduli space about M is two dimensional, we cannot infer exactly *where* to go, only the number of dimensions. Because of this, deformation theory will give us a non-zero $\mathrm{Ext}^2_A(M, M)$ meaning that there are obstructions in the deformations of A in M. In this given example, any point not on the two lines $x = 0$ and $y = 0$ is not an A-module, and is hence not in the moduli space of A. In the example we just discussed, it can seem obvious what the moduli space is, but that is generally not the case as it is risky to use intuition to conclude anything in the noncommutative plane. Therefore we must take care of not infering too much globally, when computing something locally. We will have a closer look on the calculations involved in this example in section 6.3.

Given a k-algebra A, an A-module M and the formal deformation ring $R = k[[t]]$, take the R-deformation with

$$\mu_R(a, m) = am + t\psi_1(a, m) + t^2\psi_2(a, m) + \cdots .$$

We define $\mathcal{S} = k[[\mathrm{Ext}^1_A(M, M)^*]] = k[[v_1, v_2, \cdots, v_n]]$ where $\{v_i\}_{1\le i\le n}$ is a dual basis for $\mathrm{Ext}^1_A(M, M)$. $\mathcal{S}$ is a polynomial ring in $\dim \mathrm{Ext}^1_A(M, M)$ variables. The *local moduli space* of first order is then

$$\mathcal{M}' = \{\mathcal{S} \to k\} \cong k^p \qquad \text{for } p = \dim(\mathrm{Ext}^1_A(M, M))$$

which is the family of all punctuations of $\mathcal{S}$. This is just our first candidate of the local moduli space, because it depends on there being obstructions in $\mathrm{Ext}^2_A(M, M)$ or not. There are two possibilities. Let us state the first one as a lemma:

Lemma 5.1. *If $Ext^2_A(M, M) = 0$, every infinitesimal deformation is integrable, and the local moduli space is $\mathcal{M}'$.*

Proof. From chapter 3.1 we get from associativity of the R-deformation that $\delta^1(\psi_1) = 0 \Rightarrow \psi_1 \in \mathrm{Der}(A, A)$ and $\delta^1(\psi_2) = -(\psi_1 \smile \psi_1)$. Using Lemma (2.3) we know that $(\delta^2\delta^1) = 0$ so $\mathrm{Im}(\delta^1) \subseteq \mathrm{Ker}(\delta^2)$. But $\mathrm{Ext}^2_A(M, M) = \mathrm{Ker}(\delta^2)/\mathrm{Im}(\delta^1) = 0$, implying that $\mathrm{Ker}(\delta^2) = \mathrm{Im}(\delta^1)$ and that there are no obstructions in the deformation in M. In other words, for every $\psi_i, \psi_j \in \mathrm{Im}(\delta^1)$ $i, j \in \mathbb{Z}_+$, we are sure there are no problems for later ψ_k for $k = i + j$, and the infinitesimal deformation is hence integrable. □

However if $\mathrm{Ext}^2_A(M, M) \neq 0$, there exists an element $\psi_i \smile \psi_j \in \mathrm{Ker}(\delta^2)$ with $\psi_i \smile \psi_j \notin \mathrm{Im}(\delta^1)$. This means that for some ψ_i, ψ_j, with $\psi_i \smile \psi_j \neq 0$ in $\mathrm{Ext}^2_A(M, M)$, meaning $\psi_i \smile \psi_j \notin \mathrm{Im}(\delta^1)$. This causes trouble, because we then risk choosing some ψ_i and ψ_j that satisfies their respective properties, but when combined by the cup product we don't know if $\psi_k \in \mathrm{Im}(\delta^1)$ for a $k = i + j$. To avoid this problem, we need to find these pairs and

[2]Remember that $k(0,0)$ means that the module is k and the module operation is $x \mapsto 0$ and $y \mapsto 0$

divide them out of the moduli space. Let $\mathcal{P} = \{(\psi_i, \psi_j) \mid \psi_i \smile \psi_j \neq 0\}$ and $\{w_1, \cdots, w_m\}$ be a basis for $\text{Ext}^2_A(M, M)$, then the moduli space of *quadratic* order is

$$\begin{aligned} \mathcal{M}'' &= k[[v_1, \cdots, v_n]]/(f_1, \cdots, f_m) \quad \text{where} \\ f_l &= \sum_{i,j} w_l(\psi_i \smile \psi_j) v_i v_j \end{aligned} \tag{5.1}$$

In this way we get rid of the basis vectors containing the obstructions. When $\text{Ext}^2_A(M, M) \neq 0$ we must work out the moduli space of higher and higher order, to get better and better approximation of the local moduli space. It is clear that calculating the local moduli space of high orders will need a lot of tedious calculation, but we will only concentrate on the first and second order.

5.2. **Simplifyable Modules.**

The notion of a tangent space is intuitively something local, but when working with non-commutative rings there is also possible to study tangent spaces between *two* modules M and N. This is done by simply calculating $\text{Ext}^1_A(M, N)$ instead of $\text{Ext}^1_A(M, M)$. The answer of this gives us the space containing every extension of M by N. If this space is zero, every extension of M by N is split and is hence not contained in the moduli space. If it is non-zero, there exists an indecomposable extension module E satisfying $0 \to N \to E \to M \to 0$ and we say that there is a *tangent from M to N*.

As previously stated, the moduli space consists of simple modules and non-splitting extensions. In our work of describing the moduli space, it is of interest to find out whether an extension module is *simplifyable* or not.

Definition 5.1. *Let k be an algebraically closed field. An A-module E satisfying $End_A(E) \cong k$ is simplifyable if there exists a flat family $\mathcal{E} \to T$ of A-modules with T a non-singular curve, together with a point 0 in T such that the fiber $\mathcal{E}_0 \cong E$, and such that for $t \in T$, $t \neq 0$, the fiber E_t is simple.*

In other words, E is simplifyable if it is possible to deform it into a simple module. Of course, every simple module is simplyfyable, so the interesting part is to classify the simplifyable extension modules. The following theorem gives us a necessary criterion for simplifyability.

Theorem 5.1. *Let A be an associative k-algebra and let E be an A module saisfying $End_A(E) \cong k$. Let $\mathcal{V} = Supp(E) = \{M_1, \cdots, M_r\}$ and let $\mathcal{QV}$ be the extension graph. Suppose E is simplifyable. Then there exists a complete cycle in $\mathcal{QV}$.*

Proof. [5, Thm. 4.7] □

To understand this theorem, we must first clarify some vocabulary. The *extension graph $\mathcal{QV}$* is the directed graph with the modules in $\mathcal{V}$ as vertices and the $\text{Ext}^1_A(M_i, M_j)$ as arrows for $i \neq j$. The arrow from M_i to M_j exist if and only if $\text{Ext}^1_A(M_i, M_j) \neq 0$. A *cycle* is a path starting and ending in the same vertex, and is said to be *complete* if it contains all the vertices of $\mathcal{QV}$.

If we find a non-zero $\text{Ext}^1_A(M, N)$ and $\text{Ext}^1_A(N, M)$ for two A-modules M and N, we know from theorem (5.1) that there is a good chance that the extension module E of M by N is simplifyable. In section 6, we will try to find such complete cycles for various rings in the noncommutative affine plane.

6. Calculating Local Moduli Spaces of Noncommutative Rings

We will in the following examples consider noncommutative rings denoted $k\langle x, y\rangle/(F)$ where (F) is a *quadric*.

Definition 6.1. *A quadric in the noncommutative affine plane $A = k\langle x, y\rangle$ is the zero locus of a quadratic polynomial on the form:*

$$F(x, y) = ax^2 + bxy + cyx + dy^2 + ex + fy + g$$

where $a, b, c, d, e, f, g \in k$.

By alternating (F), we can explore different kinds of structures evolving in the noncommutative case.

6.1. **A commutative example.**

As an introductary example we will look at the commutative ring $k\langle x, y\rangle/(xy - yx) := k[x, y]$.

Let $A = k[x, y]$ and $M = N = k(a, b)$. First, we need to find a projective resolution of M, so by using lemma (4.1) we start off with free A-modules:

$$0 \longleftarrow M \xleftarrow{\mu} A \xleftarrow{f} A^2 \xleftarrow{g} X$$

where f is defined by $(1, 0) \mapsto (x - a)$ and $(0, 1) \mapsto (y - b)$. Now we need to find X with a homomorphism g such that $\mathrm{Im}(g) = \mathrm{Ker}(f)$. Trying $X = A$ and setting $g = (u, v)^T$, we must solve this equation to achieve exactness

$$\begin{aligned} f \circ g = (x - a, y - b)(u, v)^T &= u(x - a) + v(y - b) = 0 \\ \Rightarrow u = y - b \ \text{ and } \ v &= -(x - a). \end{aligned}$$

Note that this answer is easy to spot, because $xy = yx$. But when (F) is different, we must potentially solve many linear equations to find f. Also noticing that g is injective, we get the following projective resolution of M:

$$0 \longleftarrow M \xleftarrow{\mu} A \xleftarrow{f} A^2 \xleftarrow{g} A \longleftarrow 0$$

To calculate $\mathrm{Ext}^1_A(M, N)$ we use the functor $\mathrm{Hom}_A(-, N)$ on this resolution to obtain:

$$\mathrm{Hom}_A(A, N) \xrightarrow{\hat{d^0}} \mathrm{Hom}_A(A^2, N) \xrightarrow{\hat{d^1}} \mathrm{Hom}_A(A, N) \xrightarrow{\hat{d^2}} \mathrm{Hom}_A(0, N)$$

where $\hat{d^0} = f \circ h$ for all $h \in \mathrm{Hom}_A(A, N)$ and $\hat{d^1} = g \circ h'$ for every $h' \in \mathrm{Hom}_A(A^2, N)$, are the induced homomorphisms of f and g. In our example $M = N = k(a, b)$, so this sequence is isomorphic to

$$k \xrightarrow{d^0} k^2 \xrightarrow{d^1} k \xrightarrow{d^2} 0$$

with $d^0 = (a - a, b - b)^T = (0, 0)^T$ and $d^1 = (b - b, -a + a) = (0, 0)$. Finally we get

$$\mathrm{Ext}^1_A(M, N) = \mathrm{Ext}^1_A(M, M) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) = k^2/0 \cong k^2$$

with basis $\{u^*, v^*\}$ and

$$\mathrm{Ext}^2_A(M, N) = \mathrm{Ext}^2_A(M, M) = \mathrm{Ker}(d^2)/\mathrm{Im}(d^1) = k/0 \cong k.$$

with basis $\{w^*\}$. Because $\mathrm{Ext}^2_A(M, M) \neq 0$, the local moduli space of M is not $\mathcal{M}'$ so we must calculate further to approximate it with $\mathcal{M}''$. We use formula (5.1) to find a polynomial

q to divide $k[[u,v]]$ with. First we find pairs (ϕ_1^u, ϕ_2^u) and (ϕ_1^v, ϕ_2^v) such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, equal] & A^2 \arrow[l, "f"'] \arrow[d, equal] \arrow[dl, "\phi_1"] & A \arrow[l, "g"'] \arrow[d, equal] \arrow[dl, "\phi_2"] \\
A & A^2 \arrow[l, "f"'] & A \arrow[l, "g"']
\end{tikzcd}$$

The pairs must then satisfy

$$\begin{aligned} &\phi_1 \circ g = f \circ \phi_2 \\ \Rightarrow\ &\phi_1 \circ (y-b, -x+a)^T = (x-a, y-b) \circ \phi_2. \end{aligned} \tag{6.1}$$

If $\phi_1^u = (1,0)$, then $\phi_2^u = (0,1)^T$. Let this pair be associated with the dual of the basis vector u^* giving $u^{**} = u$. The other solution of equation (6.1) associated with v^* is $\phi_1^v = (0,1)$ and $\phi_2^v(-1,0)^T$. We can now calculate q by

$$\begin{aligned} q &= (u \smile u)u^2 + (u \smile v)uv + (v \smile u)vu + (v \smile v)v^2 \\ &= 0 \cdot u^2 + 1 \cdot uv + (-1) \cdot vu + 0 \cdot v^2 \\ &= uv - vu \end{aligned}$$

and we can find the local moduli space of second order of A in M:

$$\mathcal{M}'' = k[[u,v]]/(uv - vu).$$

6.2. **Noncommutative partial differentiation.**

Given a ring $A = k\langle x, y\rangle/(F)$ for an arbitrary ideal (F), and A-modules M and N, it can be time consuming to find the projective resolution by solving the equations involved in the previous example. This work can very much be simplified by introducing the *partial derivative* of abstract algebra. Let the noncommutative partial differentiation of F with respect to x and (a,b) from the right be denoted by

$$D_x(F)(a,b) \in \mathrm{Hom}_k(A, A).$$

This differentiation has the following properties:

(1) $D_x(F_1 + F_2)(a,b) = D_x(F_1)(a,b) + D_x(F_2)(a,b)$ for ideals F_1 and F_2.
(2) $D_x(c)(a,b) = 0$ for $c \in k$
(3) $D_x(y)(a,b) = 0$
(4) $D_x(x)(a,b) = 1$
(5) $D_x(F_1\ F_2)(a,b) = D_x(F_1)(a,b)F_2(a,b) + F_1(x,y)D_x(F_2)(a,b)$
(Leibniz' rule) for $F_1, F_2 \in A$.

Note in the last property that the first factors in the summands are elements in A, while the second factors are evaluated in (a,b) and is hence elements in k. This is the only difference between $D_x(F)(a,b)$ and the familiar differentiation $\frac{\partial F}{\partial x}(a,b)$. These properties are of course also valid when interchanging x with y.

Let's familiarize with this new notion by differentiating $F = x^2$ and $G = x^3$.

$$\begin{aligned} D_x(F)(a,b) &= D_x(x \cdot x)(a,b) = D_x(x)(a,b)x(a,b) + xD_x(x)(a,b) \\ &= 1\ a + x = x + a \\ D_x(G)(a,b) &= D_x(x^2 \cdot x)(a,b) = D_x(x^2)(a,b)x(a) + x^2(x)D_x(x)(a) \\ &= (a+x)a + x^2 \cdot 1 = x^2 + ax + a^2 \end{aligned}$$

Remark: If we further evaluate this these expressions in (a,b), we get $2a$ and $3a^2$, which is the anticipated result from differentiating in analysis.

Because we will focus on quadrics in this thesis, we quickly calculate the following :

$$\begin{aligned} &D_x(xy)(a,b) = 1 \cdot b + x \cdot 0 = b \qquad &&D_y(xy)(a,b) = x \\ &D_x(yx)(a,b) = 0 \cdot a + y \cdot 1 = y \qquad &&D_y(yx)(a,b) = b \end{aligned}$$

The reason why this differentiation is useful when calculating Ext, becomes apparent in the discussion of the next lemma.

Lemma 6.1. *Let* $A = k\langle x_1, x_2, \cdots, x_n\rangle$ *and* $F \in A$ *such that for* $\underline{a} = (a_1, a_2, \cdots, a_n)$ *and* $F(\underline{a}) = 0$. *Then*

$$F = D_{x_1}F(\underline{a})(x_1 - a_1) + D_{x_2}(\underline{a})(x_2 - a_2) + \cdots + D_{x_n}F(\underline{a})(x_n - a_n)$$

Proof. Because $F(\underline{a}) = 0$, there exists $f_1, \cdots, f_n \in A$ such that

$$F = f_1(x_1 - a_1) + f_2(x_2 - a_2) + \cdots + f_n(x_n - a_n).$$

Now for every $i \in \{1, \cdots, n\}$ we get from the Leibniz' rule that

$$\begin{aligned} D_{x_i}F(\underline{a}) &= \Big(D_{x_i}(f_1)(\underline{a})(a_1 - a_1) + f_1 D_{x_i}(x_1 - a_1)(\underline{a})\Big) + \cdots \\ &\quad + \Big(D_{x_i}(f_i)(\underline{a})(a_i - a_i) + f_i D_{x_i}(x_i - a_i)(\underline{a})\Big) + \cdots \\ &\quad + \Big(D_{x_n}(f_n)(\underline{a})(a_n - a_n) + f_n D_{x_n}(x_n - a_n)(\underline{a})\Big) \\ &= (0 + 0) + \cdots + (0 + f_i) + \cdots + (0 + 0) \\ &= f_i \end{aligned}$$

Hence $D_{x_i}(F)(\underline{a}) = f_i \quad \forall i \in \{1, \cdots, n\}$ ☐

This means that finding Ext of any ring $A = k\langle x_1, x_2, \cdots, x_n\rangle/(F)$ can easily be determined by partial differentiation. For example for $M = k$ where $x_i \mapsto a_i$, the resolution of M is always

$$0 \longleftarrow M \xleftarrow{\mu} A \xleftarrow{f} A^n \xleftarrow{g} A \longleftarrow 0$$

where μ is the module multiplication and

$$f = (x_1 - a_1, \cdots, x_n - a_n)$$
$$g = \begin{pmatrix} D_{x_1}(F)(\underline{a}) \\ D_{x_2}(F)(\underline{a}) \\ \vdots \\ D_{x_n}(F)(\underline{a}) \end{pmatrix}$$

Because every A-module $M = k(\underline{a})$ must satisfy $F(\underline{a}) = 0$, we get from lemma (6.1) that $\mathrm{Ker}(f) = \mathrm{Im}(g)$. The module operation μ is of course surjective and $\mathrm{Ker}(g) = 0$, so by this clever way of finding g the sequence will always be exact, and hence be a resolution. Let $\underline{b} = (b_1, b_2, \cdots, b_n)$ and $N = k(\underline{b})$. We use the functor $\mathrm{Hom}_A(-, N)$ on the projective resolution above:

$$k \xrightarrow{d^0} k^n \xrightarrow{d^1} k \xrightarrow{d^2} 0$$

where

$$d^0 = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}$$
$$d^1 = \Big((\underline{b})D_{x_1}(F)(\underline{a}), \cdots, (\underline{b})D_{x_n}(F)(\underline{a})\Big).$$

Here $(\underline{b})D_{x_i}(F)(\underline{a}) \in \mathrm{Hom}_k(A, k)$ means $D_{x_i}(F)(\underline{a})$ evaluated in $\underline{b}$. Let us now see some examples where we use this type of derivation to find a projective resolution of A.

6.3. **Example:** $F = xy$.

Let $A = k\langle x, y\rangle/(xy)$ and $M = k(a, b)$. Then we find the following projective resolution of M.

$$0 \longleftarrow M \xleftarrow{\mu} A \xleftarrow{f} A^2 \xleftarrow{g} A \longleftarrow 0$$

where

$$f = (x - a, y - b)$$
$$g = \begin{pmatrix} D_x(F)(a,b) \\ D_y(F)(a,b) \end{pmatrix} = \begin{pmatrix} b \\ x \end{pmatrix}$$

We quickly check that $\mathrm{Im}(g) = \mathrm{Ker}(f)$

$$f \circ g = (x - a, y - b) \cdot \begin{pmatrix} b \\ x \end{pmatrix} = bx - ba + xy - xb$$
$$= bx - bx - ab + xy = 0$$

Applying $\mathrm{Hom}_A(-, N)$ on the projective resolution yields

$$k \xrightarrow{d^0} k^2 \xrightarrow{d^1} k \xrightarrow{d^2} 0$$

with

$$d^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$d^1 = (b, \quad a).$$

If $(a, b) \neq (0, 0)$, we get

$$\mathrm{Ext}^1_A(M, M) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong k/0 \cong k$$
$$\mathrm{Ext}^2_A(M, M) = \mathrm{Ker}(d^2)/\mathrm{Im}(d^1) \cong k/k \cong 0,$$

and for $(a, b) = (0, 0)$

$$\mathrm{Ext}^1_A(M, M) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong k^2/0 \cong k^2$$
$$\mathrm{Ext}^2_A(M, M) = \mathrm{Ker}(d^2)/\mathrm{Im}(d^1) \cong k/0 \cong k$$

This means that the local moduli space of A in M for $(a, b) \neq (0, 0)$ is $\mathcal{M}' \cong k[[u]]$, because every infinitesimal deformation is integrable.
However for $(a, b) = (0, 0)$, $k[[u, v]]/(q)$ we use formula (5.1) to determine q by the same process as in the commutative example. We find pairs of morphisms satisfying

$$\phi_1 \circ g = f \circ \phi_2$$
$$\Rightarrow \phi_1 \circ (0, x)^T = (x, y) \circ \phi_2.$$

We easily find $\phi_1^u = (0, 1)$, $\phi_2^u = (1, 0)^T$ and $\phi_1^v = (1, 0)$, $\phi_2^v = (0, 0)^T$ yielding

$$q = (u \smile u)u^2 + (u \smile v)uv + (v \smile u)vu + (v \smile v)v^2$$
$$= 0 \cdot u^2 + 1 \cdot uv + 0 \cdot vu + 0 \cdot v^2$$
$$= uv$$
$$\Rightarrow \mathcal{M}'' = k[[u, v]]/(uv)$$

We have now looked at the local tangent space of a module M, but as mentioned earlier, there is also possible to calculate the tangent space *between* two modules M and N. This is a somewhat unfamiliar notion, as a tangent space usually is a property in a single point. This is just one example of how counter-intuitive the non-commutative plane behaves.

Let A be as in the previous example, with $M = k(1, 0)$ and $N = k(0, 1)$. Note that both M and N are A-modules as they are a solution of $F = 0$. To find this tangent space, we do the exact same calculations as before with the change that we will have a different result after using the functor $\mathrm{Hom}_A(-,N)$. In this case we get

$$\begin{tikzcd} 0 & k \arrow[l] & A \arrow[l, "\mu"'] \arrow[d, "\mu'"] & A^2 \arrow[l, "f"'] & A \arrow[l, "g"'] & 0 \arrow[l] \\ & & k \arrow[r, "d^0"] & k^2 \arrow[r, "d^1"] & k \arrow[r, "d^2"] & 0 \end{tikzcd}$$

with μ and μ' being the module multiplication of respectively M and N and

$$f = (x-1,\ y) \qquad d^1 = (0,0)$$
$$g = \begin{pmatrix} 0 \\ x \end{pmatrix} \qquad d^0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We can now see that

$$\mathrm{Ext}^1_A(M,N) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong k^2/k \cong k$$

which tells us that the tangent space of M and N is one dimensional. Loosely speaking, we can take an infinitesimal step from $M = k(1,0)$ to $N = k(0,1)$. However, the same calculation interchanging M and N will give $\mathrm{Ext}^1_A(N,M) \cong 0$. This indicates that even though there exists an arrow from M to N in the extension graph of these modules, there is no cycle because we lack a non-zero extension from N to M. From theorem (5.1) we then know that there are no simplifyable extensions of M by N.

6.4. **Example:** $F = x^2 + y^2 - \lambda^2$.

Let $A = k\langle x,y\rangle/(x^2+y^2-\lambda^2)$ and $M = k(a,b)$. We find again a projective resolution of M as following:

$$0 \longleftarrow M \xleftarrow{\mu} A \xleftarrow{f} A^2 \xleftarrow{g} A \longleftarrow 0$$

where

$$f = (x-a, y-b)$$
$$g = \begin{pmatrix} D_x(F)(a,b) \\ D_y(F)(a,b) \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$$

We can check that g indeed is the function we are looking for by calculating

$$\begin{aligned} f \circ g &= (x+a)(x-a) + (y+b)(y-b) \\ &= x^2 - a^2 + y^2 - b^2 \\ &= x^2 + y^2 - a^2 - b^2 + \lambda^2 - \lambda^2 \\ &= (x^2 + y^2 + \lambda^2) - (a^2 + b^2 + \lambda^2) = 0. \end{aligned}$$

$\mathrm{Hom}_A(-,N)$ on the projective resolution gives

$$k \xrightarrow{d^0} k^2 \xrightarrow{d^1} k \xrightarrow{d^2} 0$$

with

$$d^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$d^1 = (2a,\ \ 2b).$$

If $(a,b) \neq (0,0)$, we get

$$\mathrm{Ext}^1_A(M,M) = \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong k/0 \cong k$$
$$\mathrm{Ext}^2_A(M,M) = \mathrm{Ker}(d^2)/\mathrm{Im}(d^1) \cong k/k \cong 0,$$

and for $(a,b) = (0,0)$

$$\begin{aligned}\mathrm{Ext}^1_A(M,M) =& \mathrm{Ker}(d^1)/\mathrm{Im}(d^0) \cong k^2/0 \cong k^2\\ \mathrm{Ext}^2_A(M,M) =& \mathrm{Ker}(d^2)/\mathrm{Im}(d^1) \cong k/0 \cong k\end{aligned}$$

which is the same result as in the previous example. To find the moduli space of quadratic order in $(a,b) = 0$, we must find pairs of morphisms satisfying

$$\begin{aligned}\phi_1 \circ g &= f \circ \phi_2\\ \Rightarrow \phi_1 \circ (0,x)^T &= (x,y) \circ \phi_2.\end{aligned}$$

and we easily find $\phi_1^u = (1,0)$, $\phi_2^u = (1,0)^T$ and $\phi_1^v = (1,0)$, $\phi_2^v = (1,0)^T$ yielding

$$\begin{aligned}q &= (u \smile u)u^2 + (u \smile v)uv + (v \smile u)vu + (v \smile v)v^2\\ &= 1 \cdot u^2 + 0 \cdot uv + 0 \cdot vu + 1 \cdot v^2\\ &= u^2 + v^2\\ \Rightarrow \mathcal{M}'' &= k[[u,v]]/(u^2+v^2)\end{aligned}$$

In this ring, we can find A-modules M and N such that both $\mathrm{Ext}^1_A(M,N)$ and $\mathrm{Ext}^1_A(N,M)$ are non-zero, satisfying the required property for simplifyability. Let $M = k(a,b)$ and $N = k(a',b')$ with $a,b,a',b' \in k$ satisfying $a^2+b^2 = \lambda^2$ and $(a')^2 + (b')^2 = \lambda^2$. With our familiar procedure of calculating $\mathrm{Ext}^1_A(M,N)$ and $\mathrm{Ext}^1_A(N,M)$ we find

$$\begin{tikzcd} 0 & M \arrow[l] & A \arrow[l, "\mu"'] \arrow[d, "\mu'"] & A^2 \arrow[l, "f"'] & A \arrow[l, "g"'] & 0 \arrow[l] \\ & & k \arrow[r, "d^0"] & k^2 \arrow[r, "d^1"] & k \arrow[r, "d^2"] & 0 \end{tikzcd}$$

$$\begin{aligned}f &= (x-a,\ y-b) & d^1 &= (a'+a, b'+b)\\ g &= \begin{pmatrix} x+a \\ y+b \end{pmatrix} & d^0 &= \begin{pmatrix} a'-a \\ b'-b \end{pmatrix}\end{aligned}$$

and

$$\begin{tikzcd} 0 & N \arrow[l] & A \arrow[l, "\mu'"'] \arrow[d, "\mu"] & A^2 \arrow[l, "f'"'] & A \arrow[l, "g'"'] & 0 \arrow[l] \\ & & k \arrow[r, "d'^0"] & k^2 \arrow[r, "d'^1"] & k \arrow[r, "d'^2"] & 0 \end{tikzcd}$$

$$\begin{aligned}f' &= (x-a',\ y-b') & d'^1 &= (a+a', b+b')\\ g' &= \begin{pmatrix} x+a' \\ y+b' \end{pmatrix} & d'^0 &= \begin{pmatrix} a-a' \\ b-b' \end{pmatrix}\end{aligned}$$

It is not hard to see that to get non-zero $\mathrm{Ext}^1_A(M,N)$ and $\mathrm{Ext}^1_A(N,M)$, we must have $N = (-a,-b)$. We conclude that

Proposition 1. *Every pair of antipodal points M and N on a circle with radius λ in the noncommutative affine plane $k\langle x,y\rangle$ satisfies $Ext^1_A(M,N) \cong Ext^1_A(N,M) \cong k$ and produces a cycle in the induced extension graph.*

6.5. **For which pairs M and N of one dimensional A-modules, will $\mathbf{Ext}_A^1(M,N) \neq 0$?**

After looking at these examples, it is tempting to take a closer look at a more general approach. We want to see if we can infer some properties of the quadric by taking some convenient assumptions. Let k be an algebraically closed field and F be a quadric in the noncommutative affine plane $k\langle x,y\rangle$, then we know from definition (6.1) that

$$F(x,y) = ax^2 + bxy + cyx + dy^2 + ex + fy + g$$

where $a,b,c,d,e,f,g \in k$. To make things easier to work with we can try to make the list of the seven unknown variables shorter, by introducing some linear transformations and making some convenient assumptions. We assume that $a \neq 0$, because our goal is to examplify the general quadric and it will not give any additional interesting structure to include $a = 0$. When $a \neq 0$ we might as well assume $a = 1$ by rescaling and renaming the other variables. With some equation solving we see that the transformation

$$\begin{aligned} x &\mapsto fx + (bf - de)y \\ y &\mapsto -ex + (ce - f)y \end{aligned}$$

makes the xy- and x term vanish, so after scaling and renaming we are left with

$$F \cong x^2 + cyx + dy^2 + fy + g.$$

Assuming that $4d - c^2 \neq 0$, the constant term g vanishes after the transformation:

$$x \mapsto x - \frac{-\frac{cf}{d-\frac{c^2}{4}} \pm c\sqrt{\frac{f^2}{(d-\frac{c^2}{4})^2} - \frac{4g}{d-\frac{c^2}{4}}}}{4}$$

$$y \mapsto y + \frac{-\frac{f}{d-\frac{c^2}{4}} \pm \sqrt{\frac{f^2}{(d-\frac{c^2}{4})^2} - \frac{4g}{d-\frac{c^2}{4}}}}{2}.$$

This means that for any quadric F in $k\langle x,y\rangle$ satisfying these assumptions, we can after some linear transformations express the same structure with fewer variables:

$$F \cong x^2 + cyx + dy^2 + fy$$

Let $A = k\langle x,y\rangle/(F)$. We want to find pairs of 1-dimensional modules (M,N), such that $\mathrm{Ext}_A^1(M,N) \neq 0$. So given $M = k(m,n)$, we need to find every $N = k(m',n')$ yielding a non-zero Ext. We find a projective resolution of M and use the functor $\mathrm{Hom}_A(-,N)$ as before:

$$\begin{tikzcd} 0 & k \arrow[l] & A \arrow[l, "\mu"'] \arrow[d, "\mu'"] & A^2 \arrow[l, "f"'] & A \arrow[l, "g"'] & 0 \arrow[l] \\ & & k \arrow[r, "d^0"] & k^2 \arrow[r, "d^1"] & k \arrow[r, "d^2"] & 0 \end{tikzcd}$$

with μ and μ' being the module operation of respectively M and N, and

$$\begin{aligned} f &= (x-m,\ y-n) \\ g &= \begin{pmatrix} D_x(F)(m,n) \\ D_y(F)(m,n) \end{pmatrix} \\ d^0 &= \begin{pmatrix} m'-m \\ n'-n \end{pmatrix} \\ d^1 &= \Big(D_x(F)(m,n)(m',n'), \quad D_y(F)(m,n)(m',n') \Big) \end{aligned}$$

By using the noncommutative partial differentiation we find

$$
\begin{aligned}
D_x(F)(m,n) &= x + m + cy, & D_x(F)(m,n)(m',n') &= m' + m + cn' \\
D_y(F)(m,n) &= cm + d(y+n) + f, & D_y(F)(m,n)(m',n') &= cm + d(n'+n) + f
\end{aligned}
$$

We already know that $d^0 \neq 0$, so for $\mathrm{Ext}^1_A(M,N)$ to be non-zero we need $\mathrm{Ker}(d^1) \ncong \mathrm{Im}(d^0)$, implying

$$D_x(F)(m,n)(m',n') = D_y(F)(m,n)(m',n') = 0$$

which translates into the two linear equations

(1) $m + m' + cn' = 0$
(2) $cm + d(n+n') + f = 0$.

Assuming $d \neq 0$ we can now solve these linear equations to get expressions for m' and n' in terms of m and n:

$$
\begin{aligned}
m' &= \frac{cf + c^2 m}{d} - m + cn \\
n' &= \frac{-f - cm}{d} - n.
\end{aligned}
$$

giving us N in terms of M.

We can now infer from the above discussion that the structure of most quadrics can be described by the equation $F \cong x^2 + cyx + dy^2 + fy$, and that in the ring $A = k\langle x,y\rangle/(F)$ the A-module N that produces a non-zero $\mathrm{Ext}^1_A(M,N)$ with $M = k(m,n)$ is on the form:

$$N = k\left(\frac{cf + c^2 m}{d} - m + cn, \quad \frac{-f - cm}{d} - n\right)$$

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